Total correlations as fully additive entanglement monotones

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We generalize the strategy presented in Refs. [1, 2], and propose general conditions for a measure of total correlations to be an entanglement monotone using its pure (and mixed) convex-roof extension. In so doing, we derive crucial theorems and propose a concrete candidate for a total correlations measure which is a fully additive entanglement monotone.

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I. INTRODUCTION

Since quantum entanglement was recognized as a physical resource [3], the task of quantifying the amount of entanglement present in a given multipartite quantum state became a subject of outstanding importance [3]. This task is certainly not trivial, and even for the bipartite case there are issues which are not completely understood [4, 5]. The problem gets even more involved in the multi-qudit scenario, where many more challenges arise due to the structure of the product of Hilbert spaces [6].

In terms of the capability of certain states to perform a given computational task, the problem can be posed as to how to establish a hierarchy for the degree of entanglement of the given states. However, the concept of 'more entangled' in this particular sense is quite relative as it would depend on the task we want to perform. An alternative approach to follow is that related to the development of axiomatic entanglement measures [7, 8], which tries to avoid the above feature. It establishes some basic properties that are to be satisfied, and introduces some others that are desired, e.g., additivity. The 'additivity problem' is of relevance as it is related to various quantum information features such as channel capacity [5].

In this work, we propose a general approach to multipartite entanglement measures starting from the concept of total correlations, seeking to generalize the entanglement of formation, which is in essence the quantum mutual information and its pure convex-roof construction [9]. We do this by establishing general conditions on total correlations functions and their convex-roof constructions so that they are additive entanglement measures. Surprisingly, it all comes down to one property which has only very recently been studied [10]. We then propose a particular candidate for a total correlations measure which captures all possible types of correlations and which is consistent with additivity and strong super additivity.

The paper is organized as follows. In the first section we introduce our scheme and derive some properties for a previously reported measure [1, 2] that is of interest to this work. Then, we generalize the argument and obtain general conditions and properties for total correlations measures that lead to an accurate quantification of entanglement, following the example of the Entanglement of Formation [4]. We give a specific candidate for the strategy presented and show that it satisfies the mentioned properties, thus obtaining a measure of entanglement that is fully additive.

II. MOTIVATION

We start by proposing a multipartite entanglement measure for pure states. Let's say we have an N-qudit pure state, then we define the entanglement measure as

$$\mathcal{M}_{\mathcal{P}} = \sum_{(A,B)} \mathcal{P}(A,B) \ , \tag{1}$$

where the sum is intended over all non-equivalent choices of indexes (A,B), and $\mathcal{P}$ is a probe quantity measuring the non-factorizability of a two qudit density matrix ($\mathcal{P}(\rho_{AB}) = 0$ iff $\rho = \rho_A \otimes \rho_B$). For the moment we won't give more details on the form of $\mathcal{P}$ as we want to emphasize the freedom we have on its choice. Our main candidate is the mutual information [11]

$$\mathcal{P} = \tfrac{1}{2} I(A:B) \equiv \mathcal{F}r \ . \tag{2}$$

Note that for two qudits the measure is exactly the Entanglement of Formation. Next we extend the measure to mixed states using the convex-roof construction [9]; thus

$$\mathcal{M}_{\mathcal{P}} = \min \sum p_i \sum_{(A,B)} \mathcal{P}(A,B)(\rho^i) \ , \tag{3}$$

where the minimization is over every possible decomposition on pure states.

A. Benefits of the strategy

We now expand on the benefits of the so-called pairwise strategy. We introduce an alternate way of finding the minimizing decomposition.

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Pairwise minimizing decomposition.— It is mainly motivated by the following reasoning

$$\mathcal{M}(\rho) = \min \sum p_l \mathcal{M}(\rho_l) = \min \sum_l p_l \Big(\sum_{s_i \neq s_j} \mathcal{P}(\rho^{(l)}_{s_i s_j} \Big) = \min \sum_{s_i \neq s_j} \sum_l p_l \mathcal{P}(\rho^{(l)}_{s_i s_j}) \ , \tag{4}$$

which implies that the minimization condition is equivalent to minimizing the value of $\mathcal{P}$ for each two qudit reduced density matrix for decompositions over mixed density matrices, that is $\mathcal{P}(\rho_{s_i s_j}) = \min \sum_l q_l \mathcal{P}(\rho^{(l)}_{s_i s_j})$, where $\rho^{(l)}_{s_i s_j}$ may be mixed, with three simultaneous constraints: i) All reduced density matrices must have the same coefficients, if the two qubit reduced density matrices are minimized by $\rho_{s_i s_j} = \sum_l q_l^{(s_i s_j)} \rho^{(l)}_{s_i s_j}$ then $q_l^{(s_i s_j)} = f_l$ for all pairs $(s_i s_j)$, ii) The set of two qubit reduced density matrices $\rho^{(l)}_{s_i s_j}$ correspond to an n-partite pure density matrix ρ_l for each l, and iii) ρ is expanded by $\sum p_l \rho_l$. This definition is consistent when ρ is a pure density matrix: condition iii) requires that there is only one non vanishing q_l, thus automatically guaranteeing conditions i) and ii) and reducing to our previously defined measure. Thus, we are re-establishing the problem of finding the N-qubit matrix decomposition, with the one of finding $C_2^N \equiv N!/(N-2)!2!$ two qubit minimizing decompositions (for all possible two qudit reduced density matrices). Issues regarding computability in this way shall be addressed later, when we discuss the details of the advantages posed by this procedure.

Theorem II.1 *The entanglement measure $\mathcal{M}_{\mathcal{P}}$ is fully additive, i.e.*

$$\mathcal{E}(\sigma \otimes \eta) = \mathcal{E}(\sigma) + \mathcal{E}(\eta) \ , \tag{5}$$

provided that

$$\sum_{A,B} \mathcal{P}(\rho_{AB}) \geq \min \sum p_i \sum_{A,B} \mathcal{P}(\rho_{AB})^{(i)} \ . \tag{6}$$

Proof We rely on the pairwise minimizing condition. Consider two generic m and $N-m$ qudit density matrices, η and σ; hence there exists a bifactorizable decomposition of the form $\eta \otimes \sigma = p_i \eta^i \otimes q_i \sigma^i$. The proof assumes that we have a generic, non-bifactorizable, minimizing decompostion. We show that the bifactorizable decomposition has a lower value of entanglement following the convex-roof construction procedure. For concreteness we show it here for the $N = 4, m = 2$ case, but the argument can be easily extrapolated to the multipartite case.

If we have a non-bifactorizable decomposition of the form $\rho = \sum p_i \sigma^i$, then we have that there are non-vanishing pairwise minimizing decompositions for ρ_{12}, ρ_{13}, ρ_{14}, ρ_{23}, ρ_{24} and ρ_{34}, with values denoted as $f(\rho_{AB})$. On the other hand, a bifactorizable decomposition would have other values for their minimizing decomposition, namely $g(\rho_{AB})$, and in particular some of them vanish, $g(\rho_{13}) = g(\rho_{14}) = g(\rho_{23}) = g(\rho_{24}) = 0$, which implies that $g(\rho_{13}) \leq f(\rho_{13})$ and similarly for the $(1,4)$, $(2,3)$ and $(2,4)$ pairs.

The proof would be complete if we demand that $g(\rho_{12}) \leq f(\rho_{12})$ and $g(\rho_{34}) \leq f(\rho_{34})$. This is equivalent to demanding that the lowest value achieved by any decomposition is on pure state matrices, namely on a decomposition $\rho_{AB} = \sum_\alpha p_\alpha \rho^\alpha_{AB}$, where ρ^α_{AB} is a pure density matrix. This formalizes to request that

$$\mathcal{P}(\tilde{\rho}_{AB}) \geq \min \sum p_i \mathcal{P}(\tilde{\rho}_{AB})^{(i)} \ , \tag{7}$$

as claimed. The extension of the argument to more qudits is straightforward, and would leave us with the condition Eq. (6), where the minimization is intended over every possible decomposition on pure states. Note that if this is true then any decomposition on mixed states will yield a higher value of entanglement. ∎

In a similar way, strong super additivity can also be demonstrated provided that Eq. (6) is satisfied [2].

B. The mutual information

We now explore the particular case of the mutual information $\mathcal{P} = \frac{1}{2} I(A:B)$. This is a measure of total correlations, so it vanishes iff $\rho_{AB} = \rho_A \otimes \rho_B$, and, even more, it is always greater than the quantum correlation, as measured by the entanglement of formation, i.e.

$$\tfrac{1}{2} I(A:B)(\rho_{AB}) \geq \min \sum p_i \tfrac{1}{2} \mathcal{I}(A:B)(\rho_{AB})^{(i)} \ .$$

We conjecture that this is also true for the multipartite case, as in this case $\sum_{A,B} I(A:B) = 0$ iff $I(A:B) = 0$ for all (A,B), that is, if the state is fully factorizable (not necessarily fully separable; note that factorizability and separability are equivalent concepts only in the case of pure states).

The measure can be interpreted in terms of the amount of information we get about each qudit after we measure one of them. We must have this in mind when interpreting the properties that follow, as in general we can build different measures which capture different aspects of multipartite entanglement.

C. Properties

1. Normalization

The defined measure is normalized to $\mathcal{M} \leq (2 - \delta_{2,N})^{-1} C_2^N \log_2 d$. First note that the inclusion of the δ function holds only for the two qubit case, which is easily seen to be bounded by $\log_2 d$, as it is essentially $\frac{1}{2} I(A:B)$. Second, we give the proof for the cases of three, four and five qudits. Similar inequalities can be tailored for $N > 5$ qudits. The following theorem holds for the general case of d level systems (qudits).

Theorem II.2 *For an N-qudits quantum state, $\mathcal{M}_{\mathcal{F}r}$ is normalized to $(2-\delta_{2,N})^{-1}C_2^N \log_2 d$, where $\mathcal{F}r$ denotes the von Neumann's mutual information.*

Proof For $\mathcal{F}r(A,B) = \frac{1}{2}[S(\rho_A) + S(\rho_B) - S(\rho_{AB})]$, where $S(\rho)$ is the von Neumann's entropy, we can prove that the measure is indeed normalized in the following way. To simplify the notation, we denote $S(X) \to X$. The von Neumann's entropy strong sub additivity reads [12]

$$XYZ \le XY + YZ - Y \ . \tag{8}$$

We shall use this inequality for different partitions throughout the paper.

The proof for the three qudit case, say ρ_{ABC}, is trivial. By using $S(AB) = S(C)$, we get that $\mathcal{M} = \frac{1}{2}(A+B+C) \le \frac{1}{2}\log_2 d$. For the four qubit case, using von Neumann's entropy strong subadditivity and assignations of $(X,Y,Z) = \{(B,A,C);(B,D,A);(B,C,D)\}$ it follows that

$$\begin{aligned}\mathcal{M}_{\mathcal{F}r} &\le (3B + 2A + 2C + 2D - BAC - BDA - BCD) \\ &= 3B + A + C + D \le 6\log_2 d = \tfrac{1}{2}C_2^4 \log_2 d \ ,\end{aligned}$$

where we have used that $S(\rho_i) \le \log_2 d$.

For the case $N = 5$, consider the following inequality,

$$\begin{aligned}XYZW &\le YXZ + YWZ - YZ \\ &\le XY + XZ + YW + WZ - YZ - X - W \ ,\end{aligned}$$

summing for the assignations of $(X,Y,Z,W) = \{(E,A,B,C);(E,A,C,D);(B,A,D,C);(B,A,E,D); (A,B,C,D);(A,B,D,E);(D,B,E,C);(B,C,D,E); (A,C,E,B);(A,D,E,C)\}$ we get, using that for an N-qubit pure state $S(A_1,...,A_m) = S(A_{m+1},...,A_N)$

$$6(A+B+C+D+E) - 3(AB+AC+AD+AE+BC+BD+BE+CD+CE+DE) \le 0 \ ,$$

$$12(A+B+C+D+E) - 3(AB+AC+AD+AE+BC+BD+BE+CD+CE+DE) \le 6(A+B+C+D+E) \ ,$$

Hence,

$$\mathcal{M}_{\mathcal{F}r} \le \tfrac{1}{2}C_2^5 \log_2 d \ . \tag{9}$$

For $N \ge 6$, similar inequalities hold. ∎

The above analysis shows that $\mathcal{M}_{\mathcal{P}}$ is indeed normalized, and reaches its maximum for the $|GHZ\rangle$ states, which we define as follows.

Definition II.3 (GHZ state) *A $|GHZ\rangle$ state is the state with the highest average of $\mathcal{P}$ over all possible pairs of qudits.*

Following the interpretation of the mutual information, it is interesting to see that the GHZ state is then the state for which after measuring one qudit we obtain more information about every qudit on average.

2. *Entanglement monotonicity*

The proposed entanglement measure satisfies the following: i) Separable states have no entanglement. Our measure vanishes if and only if all $\mathcal{P}(A,B) = 0$, which implies that the state is of the form $|\Psi\rangle = |\psi_1\rangle \otimes ... \otimes |\psi_N\rangle$. The general separable mixed case follows from the convex-roof construction. ii) The measure is non-vanishing if the state is entangled, as seen above. iii) Entanglement doesn't change under local unitary (LU) operations. This is evident from the respective formulae for $\mathcal{P}$. It certainly depends on the form of $\mathcal{P}$, however in our case we are resorting to von Neumann's entropies, which are LU invariants. iv) There are maximally entangled states, see Theorem (II.2). To complete the picture that supports that $\mathcal{M}_{\mathcal{P}}$ is indeed a good entanglement measure, we need that v) Entanglement must be LOCC non-increasing. We have performed numerical simulations that support this claim. Furthermore, the results of the next section show that it is in fact an entanglement monotone.

We show below the local operations and classical communication (LOCC) non-increasing character of $\mathcal{M}_{\mathcal{P}}$, thus proving that this is indeed a fully additive entanglement measure.

III. MORE ON THE STRATEGY

We now push the strategy further. We emulate the Entanglement of Formation [4] in the two qudit case and apply the same argument to the multipartite case, i.e. we find a measure of total correlations $\mathcal{T}$ and use it to quantify entanglement in pure states, and then, through its convex-roof extension $\mathcal{T}^*$, extend it to mixed states. We must then also require that the total correlations measure $\mathcal{T}$ is additive and strongly sub additive on mixed states, i.e. that the following properties hold:

ADD *Additivity.* Given two arbitrary states denoted by ρ_A and ρ_B,

$$\mathcal{T}(\rho_A \otimes \rho_B) = \mathcal{T}(\rho_A) + \mathcal{T}(\rho_B) \ . \tag{10}$$

SSA *Strong super additivity.* Given a generic N-partite state $\rho^{1,\dots,N}$,

$$\mathcal{T}(\rho^{1,\dots,N}) \geq \mathcal{T}(\rho^{1,\dots,m}) + \mathcal{T}(\rho^{m+1,\dots,N}) \ . \tag{11}$$

This is a natural condition to ask, as when we make a partition on the state we are immediately destroying correlations and thus justifying the inequality.

PCRC *Pure Convex-roof consistent.* This is the generalization of Eq. (6), and is the requirement that the convex-roof minimization is attained on decompositions over pure states, which is equivalent to

$$\mathcal{T}(\rho) \geq \mathcal{T}^*(\rho) = \min \sum p_a \mathcal{T}(\rho_a) \ , \tag{12}$$

where the minimization is intended over pure state decompositions of the state ρ.

A total correlations measure that satisfies the above conditions shall be referred to as a *complete total correlations measure*. We show next that a measure of total correlations satisfying the above mentioned conditions leads to an additive measure of entanglement provided it also satisfies the monotonicity conditions.

Theorem III.1 *Let $\mathcal{T}$ be a measure of total correlations on pure states, and let $\mathcal{T}^*$ be its pure convex-roof extension. If $\mathcal{T}$ satisfies ADD, SSA and PCRC then it is a fully additive and strongly super additive quantity. We say it is also a fully additive entanglement measure if it is also an entanglement monotone.*

Proof We consider, for concreteness, the four-partite case, but the argument can easily be extended to the N-partite case. Let's consider two two-qudit density matrices $\rho^{(1)}$ and $\rho^{(2)}$ with optimal decompositions $\rho^{(i)} = \sum p_a^{(i)} \sigma_a^{(i)}$, such that $\rho = \rho^1 \otimes \rho^2$. We first consider an arbitrary non-bifactorizable decomposition, and then we will show that it must have higher values for the convex-roof extension compared to a bifactorizable decomposition, thus showing that the bifactorizable decomposition is indeed the real minimum for the pure convex-roof construction:

$$\begin{aligned}
\mathcal{T}^*(\rho) &= \sum q_a \mathcal{T}(\rho_a^{1234}) \\
&\geq \sum q_a (\mathcal{T}(\rho_a^{12}) + \mathcal{T}(\rho_a^{34})) \qquad \text{(by SSA)} \\
&= \sum q_a (\mathcal{T}(\rho_a^{12})) + \sum q_a (\mathcal{T}(\rho_a^{34})) \\
&\geq \sum q_a (\min \sum_s u_s^{(a)} \mathcal{T}(\rho_s^{(a)12})) + \text{i.d. over } \{34\} \\
&\geq \mathcal{T}^*(\rho^1) + \mathcal{T}^*(\rho^2) \ ,
\end{aligned}$$

where the last inequality follows as the decomposition resulting of minimizing every mixed density matrix in the expansion may not be actual minimal decomposition of the complete matrix. In other words,

$$\begin{aligned}
r_1 \mathcal{T}(\eta^1) + r_2 \mathcal{T}(\eta^2) &\geq \sum r_1 (\min \sum_s u_s^{(1)} \mathcal{T}(\eta_s^{(1)})) + \text{i.d.}\{2\} \\
&\geq \mathcal{T}^*(\sum r_c \eta_c) \ .
\end{aligned}$$

Strong super additivity, i.e. $\mathcal{T}^*(\rho^{1,\dots,N}) \geq \mathcal{T}^*(\rho^{1,\dots,m}) + \mathcal{T}^*(\rho^{m+1,\dots,N})$, can be demonstrated using the same reasoning as above, but with the identifications $\rho = \rho^{1,\dots,N}$, $\rho_1 = \rho^{1,\dots,m}$ and $\rho_2 = \rho^{m+1,\dots,N}$. ∎

IV. A TOTAL CORRELATIONS FUNCTION

We first establish the basic conditions that a measure of total correlations must fulfill. We stress that currently there are only some basic conditions [14, 15] but no real consensus on more closed conditions has been achieved. The basic conditions any total correlations function must satisfy are

TCF1 Positivity: $\mathcal{T}(\rho) \geq 0$.

TCF2 It vanishes on factorizable states only: $\mathcal{T}(\rho^{1\dots N}) = 0$ iff $\rho^{1\dots N} = \rho^1 \otimes \dots \otimes \rho^N$.

TCF3 Invariance under ancillas: $\mathcal{T}(\rho \otimes (\bigotimes_i \sigma_i)) = \mathcal{T}(\rho)$.

TCF4 LU invariance.

TCF5 LO non-increasing.

We show next that given these conditions and a pure convex-roof construction, we obtain an entanglement monotone.

Theorem IV.1 *Any complete total correlations measure $\mathcal{T}$, extended to mixed states through the pure convex-roof construction $\mathcal{T}^*$ is an entanglement monotone.*

Proof We only need to prove that a measure defined in this way is an LOCC non-increasing function, as the other properties are provided by the hypothesis. In so doing, we will make use of the FLAGS conditions introduced in Ref. [13]: an entanglement measure E is a monotone iff it is a local unitary invariant and satisfies

$$E\left(\sum p_i \rho_i \otimes |i\rangle\langle i|\right) = \sum p_i E(\rho_i) \ . \tag{13}$$

To this end, we proceed in the following way. First, by convexity and TCF3, we have

$$\mathcal{T}^*(\sum p_i \rho_i \otimes |i\rangle\langle i|) \leq \sum p_i \mathcal{T}^*(\rho_i \otimes |i\rangle\langle i|) = \sum p_i \mathcal{T}^*(\rho_i) \ .$$

Now we must show that $\mathcal{T}^*(\sum p_i \rho_i \otimes |i\rangle\langle i|) \geq \sum p_i \mathcal{T}^*(\rho_i)$ to get a full inequality. To do this, we must show that the optimal decomposition of $\tilde{\rho} = \sum p_i \rho_i \otimes |i\rangle\langle i|$ is bounded

by $\sum p_i \mathcal{T}^*(\rho_i)$. Note that the above decomposition of $\tilde{\rho}$ implies that there exists a decomposition in pure states of the form

$$\rho = \sum_s \sum_i q_s p_i^{(s)} |\Psi_i^{(s)}\rangle\langle\Psi_i^{(s)}| \otimes |s\rangle\langle s| \ , \tag{14}$$

which is valid as $\sum_{s,i} q_s p_i^{(s)} = 1$. We now show that if such a decomposition exists, then it minimizes $\mathcal{T}^*(\rho)$. As in previous cases, let's assume that the minimal decomposition is given by $\rho = p_i \rho_i^{SR}$, where S may contain any number of qudits and R contains a single qudit. Then

$$\begin{aligned}
\mathcal{T}^*(\rho) &= \sum t_a \mathcal{T}(\rho_a^{SR}) && (15)\\
&\geq \sum t_a (\mathcal{T}(\rho_a^S) + \mathcal{T}(\rho_a^R)) && \text{(by SSA)}\\
&\geq \sum t_a (\mathcal{T}^*(\rho_a^S) + \mathcal{T}^*(\rho_a^R)) && \text{(by PCRC)}\\
&= \sum t_a (\mathcal{T}^*(\rho_a^S \otimes \rho_a^R)) && \text{(by ADD)}\\
&= \sum q_a \mathcal{T}^*(\rho_a^S) \ . && \text{(by TCF3)}
\end{aligned}$$

This shows that given an arbitrary decomposition with no local flags, assumed to minimize $\mathcal{T}^*$, a decomposition of the form Eq. (14), if it exists, gives an even lower value for it, as the third and fourth lines of Eqs. (15) imply, and thus showing it is the optimal decomposition. The last line of Eqs. (15) follows in virtue of the invariance under ancillas condition and proves our claim. ∎

We now turn to the question of what the minimum conditions are for a quantity $\mathcal{Q}$ defined on pure states to be an entanglement monotone. We see that we can relax the strong super additivity condition, and just require it to be SSA for pure states, $\rho^{1,\ldots,N} = |\Psi\rangle\langle\Psi|$. Furthermore, we can relax TCF2 on the correlations functions, and allow for it to vanish only on separable pure states, recalling that the convex roof extension would still vanish on mixed and pure separable states. Also, we note that if $\mathcal{Q}$ is a concave function then it is automatically a fully additive entanglement monotone. We do not want to state this as a theorem at this point, but notice that this can be concluded from the theorems and proofs given above. In this sense, it is easy to see that the Meyer-Wallach-Brennen measure $\mathcal{MW}$ [17], with its pure convex-roof extension $\mathcal{MW}^*$, is then automatically additive, strongly super additive, and an entanglement monotone, as $\mathcal{MW} = \sum S(\rho_i)$ is a concave function and thus trivially satisfies PCRC. This measure, however, has limitations at distinguishing several states, but it provides a good straightforward example and a direct evidence of the challenges involved when defining ambiguity-free multipartite entanglement quantifiers, and of the many faces of multipartite quantum correlations. We are now ready to introduce the following definition.

Definition IV.2 *A total correlations measure $\mathcal{T}$ is a coherent total correlations measure if the maximally quantum correlated state has a higher value than the maximally classically correlated state.*

Recently, mixed convex roof extensions have been considered as a minimization process over all possible decompositions [10]:

$$\mathcal{E}_\rho = \min \sum_i p_i \mathcal{E}(\rho^i) \ , \tag{16}$$

where ρ_i is a general density matrix. In this scenario, our theorems are simplified and the PCRC condition can be dropped as the mixed convex-roof trivially satisfies it, thus obtaining fully additive entanglement monotones.

With these results at hand we would now like to verify the monotonicity conditions for our previously proposed measure and also present a generalization of it.

V. BUILDING ADDITIVE ENTANGLEMENT MEASURES: A COMPLETE AND COHERENT TOTAL CORRELATIONS MEASURE

We next analyze some issues related to the monotonicity of the proposed measure $\mathcal{M}$. For this to be a monotone, Eq. (6), which is the analog of PCRC, must be satisfied. Note that although $\mathcal{M}$ is not coherent, this does not pose a setback, as this simply means that it does not fully account for the quantification of certain type of correlations which are characteristic of a total correlations measure. This doesn't mean, however, that $\mathcal{M}$ is ill-defined, as we will see below where we build a complete correlations function which satisfies PCRC.

We first build a *coherent* and *complete* total correlations measure which considers all possible correlations while maintaining additivity and strong super additivity. There are several possibilities for correlations in a state:

i) *Pairwise total correlations.* As shown in Sect. II, they are additive and strongly super additive. ii) *Bipartite correlations.* Consider an N-qubit state and an arbitrary bipartition $\mathcal{B} = S(\rho_R) + S(\rho_{N-R}) - S(\rho_N)$. This quantity is strongly super additive but not additive in general. iii) *Correlations among subsets.* This can be considered as a general case containing the bipartite correlations, and the pairwise correlations, however their sum $\mathcal{L}$, although being strongly super additive, is not additive in general. iv) *Single qubit correlations or global correlations.* We consider single qubit entropies as global correlations. The correlations for this case are given by

$$\begin{aligned}
\mathcal{O}(\rho_{1,\ldots,N}) &= \tfrac{1}{2}\left(\sum S(\rho_i) - S(\rho_{1,\ldots,N})\right)\\
&\geq \tfrac{1}{2}\left(\sum S(\rho_i) - S(\rho_{1,\ldots,m}) - S(\rho_{m+1,\ldots,N})\right)\\
&= \mathcal{O}(\rho_{1,\ldots,m}) + \mathcal{O}(\rho_{m+1,\ldots,N}) \ , && (17)
\end{aligned}$$

thus strong super additivity is guaranteed. Additivity on pure states follows analogously. This measure has been studied previously [14, 15] and it has been proven to measure the basic properties and to be coherent. It fails, however, at discriminating between $|EPR\rangle \otimes |EPR\rangle$ and $|GHZ\rangle_4$ states, so we can say again that it does not succeed at quantifying certain total correlations.

Note that i) and ii) are particular cases of the subsets case, $\mathcal{L}$, and when we sum over all possible choices of subsets we get a strongly super additive quantity as a total correlations measure (SSA); this is however not additive. To see this, we note that when any artificial bipartition is performed on an arbitrary density matrix $\rho \rightarrow \rho_P, \rho_{\bar{P}}$, $\mathcal{L}(\rho)$ contains all positive terms in $\mathcal{L}(\rho_P) + \mathcal{L}(\rho_{\bar{P}})$, thus evidencing strong super additivity. However, when there is a natural bipartition, non-trivial multiplicities of the elements of $\mathcal{L}(\rho_P)$ and $\mathcal{L}(\rho_{\bar{P}})$ appear, and so there is no additivity.

With this in mind, we now build a coherent and complete total correlations function $\mathcal{S}$ and its pure convex-roof extension $\mathcal{S}^*$, which by the theorems above is an entanglement measure provided PCRC holds, as

$$\mathcal{S} = \frac{\mathcal{O} + \mathcal{M}}{2} \ , \tag{18}$$

which is just the average of the two types of correlations which keep additivity and strong super additivity. Their sum helps us to overcome the issues they posed separately, at the cost of making the PCRC conjectured condition weaker as $\mathcal{S}-\mathcal{M}|_{\text{on pure states}} \geq \mathcal{S}-\mathcal{M}|_{\text{on mixed states}}$.

Given the bounds for each measure, it is easy to see that

$$\mathcal{S} \leq \frac{(C_2^N(2-\delta_{N,2})^{-1} + N/2)}{2} \log_2 d \ . \tag{19}$$

Note that, through its pure convex-roof extension, it reduces to the Entanglement of Formation [4] in the bipartite case. We can alternatively write our measure as

$$\mathcal{S} = \frac{\sum_{i \leq j} S(\rho_{ij} \| \rho_i \otimes \rho_j) + S(\rho_{1,\ldots,N} \| \rho_1 \otimes \ldots \otimes \rho_N)}{4} \ , \tag{20}$$

which immediately suggests the continuity of our measure. This is, however, not surprising as our measure is written up in terms of the quantum mutual information and relative entropies, which are asymptotically continuous. Also note that if two N-partite density matrices are close, then their reduced density matrices are also close. The maximum of the measure is again attained by the $|GHZ\rangle$ state.

The benefits of using $\mathcal{S}$ instead of resorting to $\mathcal{O}$ or $\mathcal{M}$ separately are as follows. We capture more types of correlations than by means of $\mathcal{O}$ alone, which has been generally used as a measure of total correlations [15]. Consider, for example, the case of the $|GHZ\rangle$ state, the cluster state [16], and the $|EPR\rangle \otimes |EPR\rangle$ state, which exhibit different types of correlations: our measure can effectively distinguish all of them. Perhaps the most notable comparison is that between the cluster state and the $|EPR\rangle \otimes |EPR\rangle$ state: $\mathcal{O}$ alone fails to distinguish among them, whilst $\mathcal{S}$ does the job due to the inclusion of the pairwise total correlations. It is easy to see that $\mathcal{S}$, by construction, captures more types of correlations, and thus is a more complete measure of total correlations. Also note that $\mathcal{S}$ considers all types of correlations consistent with additivity and strong super additivity.

A. Comparing $\mathcal{O}$ and $\mathcal{S}$

We now quantify the above established comparison between the measures $\mathcal{O}$ and $\mathcal{S}$, and analyze the behaviour of specific cases in terms of the number of particles. Before we proceed to consider the concrete cases, we note that the measure $\mathcal{O}$ is the von Neumann analog of $\mathcal{MW}$, namely the structure is the same but with von Neumann's entropy instead of the Linear entropy. Furthermore, in our multipartite entanglement measure scenario $\mathcal{O}^* = \mathcal{MW}^*$.

We next list and define some of the states we will analyze and plot below:

i) $|GHZ\rangle_N = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right)$,

ii) $|Cluster\rangle_N = \frac{1}{\sqrt{N}}\left(|0\rangle^{\otimes N} + |0\rangle^{\otimes N/2}|1\rangle^{\otimes N/2} + |1\rangle^{\otimes N/2}|0\rangle^{\otimes N/2} - |1\rangle^{\otimes N}\right)$,

iii) $|W\rangle_N = \frac{1}{\sqrt{N}}\left(|10...0\rangle + |010...0\rangle + ... + |0...01\rangle\right)$,

iv) $|\bar{W}\rangle_N = \frac{1}{\sqrt{N}}\left(|01...1\rangle + |101...1\rangle + ... + |1...10\rangle\right)$,

v) $|EPR\rangle = |GHZ\rangle_2$.

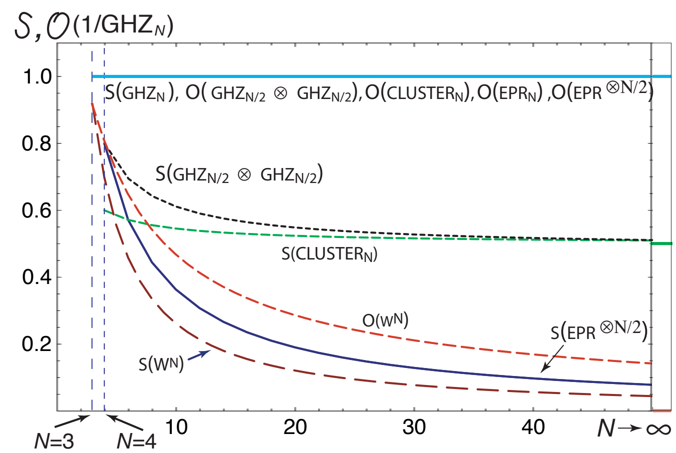

FIG. 1: *N-dependance relative entanglement comparative graph.* The entanglement of all states is compared to the entanglement of the GHZ state, thus we are plotting the relative entanglement of each state. Note how $\mathcal{O}$ fails to discriminate among several kinds of states, whilst $\mathcal{S}$ does indeed establish a hierarchy in the degree of entanglement of the different states.

First, we compare the results of the application of both measures to known states and then characterize their dependence on the size of the quantum register. This is first plotted in Fig. 1, where $\mathcal{S}$ and $\mathcal{O}$ appear evaluated for several different states as a function of the particles number N. We note that as $\mathcal{O}$ relies on single qubit von Neumann's entropies, it fails to distinguish among several states, as shown by the horizontal, solid line (in blue) of the figure. It is interesting that in the infinite qudit

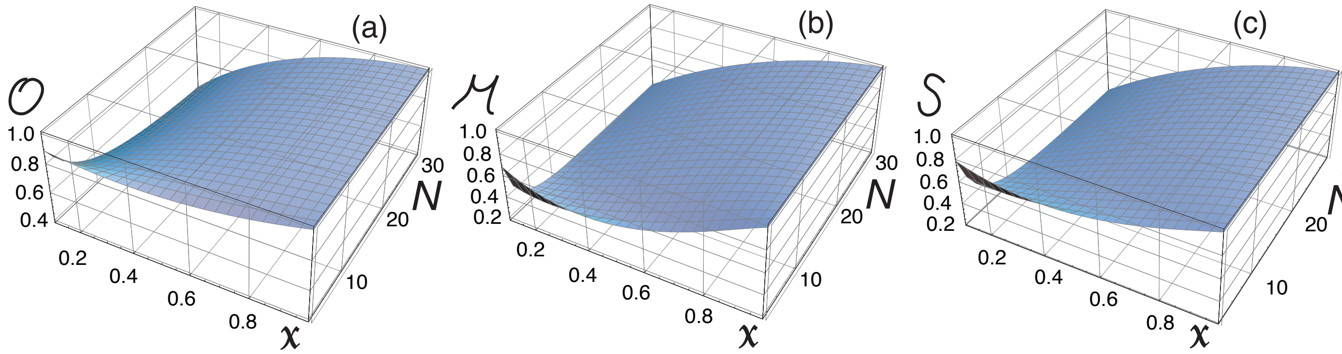

FIG. 2: a) Global, b) Pairwise, and c) Total correlations measures for the family of states $|\psi\rangle_x^{(1)} = \sqrt{x}\,|GHZ\rangle_N + \sqrt{1-x}\,|W\rangle_N$, as a function of the parameters x and N. As in Fig. 1, the graph is normalized by the value for the GHZ state.

thermodinamic limit, the cluster and the $|GHZ\rangle_{N/2}^{\otimes 2}$ have the same value of $\mathcal{O}$. This is so because their pairwise correlations structure is the same, namely the same pairwise total correlations vanish (permutations are of course analog) with different values, and the global correlations compensate this difference in such a way that they yield the same limit. The graph also shows the distinguishability advantages of our measure $\mathcal{S}$ when applied to all of the above introduced quantum states.

In the same comparative spirit, we now proceed to analyze two cases of particular interest. In the first case, that plotted in Fig. 2 for the state $|\psi\rangle_x^{(1)} = \sqrt{x}\,|GHZ\rangle_N + \sqrt{1-x}\,|W\rangle_N$ as a function of the parameters x and N, there is no major difference between the global ($\mathcal{O}$), pairwise ($\mathcal{M}$) and the total measure ($\mathcal{S}$), only a small quantitative discrepancy in their values as a function of x and N. Thus, we note that for some states the behaviour of the different measures is quite similar, i.e. for some states the contribution due to the pairwise correlations is not very significant.

The next case, however, evidences the existence of states for which the pairwise contributions become of particular relevance. This case poses two main features: i) pairwise correlations become important, and ii) entanglement or total correlations raise with the number of qubits for a range of x values, as can be seen in Fig. 3. In the thermodinamic limit, the state $|\psi\rangle_{x=1/2}^{(2)} = \frac{1}{\sqrt{2}}\left(|W\rangle_N + |\bar{W}\rangle_N\right)$ has the same entanglement as the GHZ state. This is very interesting as, in principle, only in the thermodinamic limit would one have enough degrees of freedom to perform local unitary operations to transform one state into the other thus justifying the equality.

We have then shown how total correlations measures can generate fully additive entanglement monotones using its convex-roof extension to mixed states. In so doing, we have also found the relevant conditions for the case of pure or mixed convex-roof extensions.

As a main result, we would like to stress that, using the mixed convex-roof extension, *the proposed total correlations measure $\mathcal{S}$ is a complete coherent total correlations measure as well as a fully additive entanglement monotone*.

We anticipate, as a perspective, that the results provided here would allow the construction of a proof of the long conjectured additivity of the Entanglement of Formation [5]. The proof of such a conjecture has deep implications on the Holevo bound and quantum channel additivity, among others results in quantum information theory [18].

VI. CONCLUSIONS

We have proposed a strategy for quantifying entanglement in the multipartite case, based on measures of total correlations and its pure convex-roof extension. Within a natural scenario, we have demonstrated that these total correlations measures are entanglement monotones. Furthermore, we have proposed a specific quantity to simultaneously fulfill the role of a total correlations measure and a fully additive entanglement monotone.

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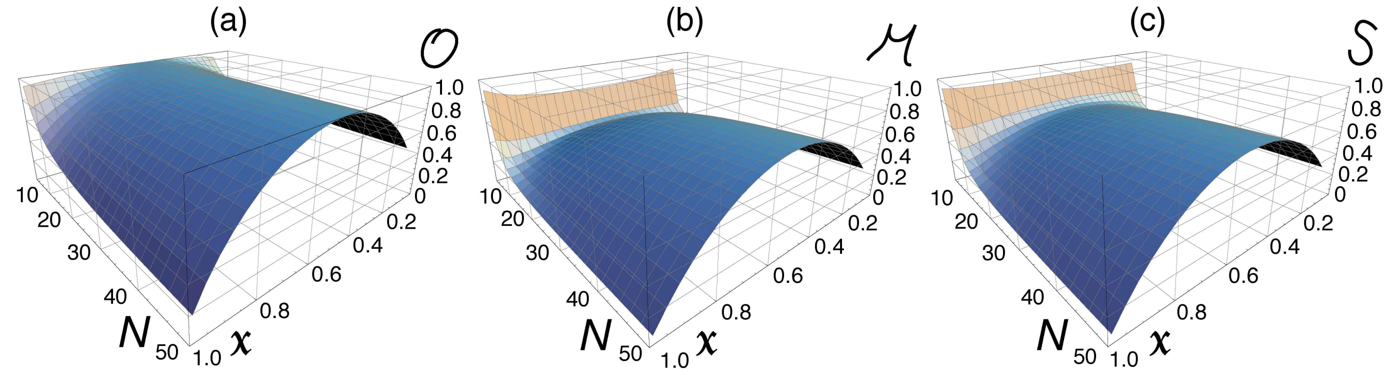

FIG. 3: a) Global, b) Pairwise, and c) Total correlations measures for the family of states $|\psi\rangle_x^{(2)} = \sqrt{x}\,|W\rangle_N + \sqrt{1-x}\,|\bar{W}\rangle_N$, as a function of the parameters x and N. As in Fig. 1, the graph is normalized by the value for the GHZ state.

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